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The Yang–Lee distribution of zeros for a classical one-dimensional fluid

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Abstract. The interaction potential is assumed to satisfy $\varphi(r) = +\infty$ if $r < a$ and $\varphi(r) = 0$ if $r > 2a$, where a is a constant greater than zero, so that only nearest neighbours can interact. At any fixed temperature T let $kT\pi(z)$ be the thermodynamic pressure at fugacity $z \geqslant 0$, as calculated from the equation of state. Let $\Pi(z)$ be the complete analytic function obtained by analytic continuation of $\pi(z)$ into the complex z plane, and G be the set of values of z for which one branch of $\Pi(z)$, say $\Pi_{\max}(z)$, is regular and has a larger real part than all the others. It is proved that, in the limit where the length L of the system tends to infinity, the zeros of the grand partition function $\Xi(z, L)$ approach a point set Z which consists of analytic arcs and is the complement of G. It is also proved that G is simply connected, that

$$\lim_{L\to\infty} \ln|\Xi(z, L)| = \operatorname{Re}\Pi_{\max}(z)$$

for all z in G, and that the limiting line density of zeros of Ξ along any arc of Z (each zero being given the weight L^{-1}) is $(2\pi)^{-1}$ times the discontinuity in $\operatorname{Im}\pi_{\max}(z)$ across the arc. As an illustration, (a result of Hemmer *et al.*), that Z is $-\infty < z \leqslant -1/ea$ for the hard-rod system is confirmed rigorously.

1. Introduction

Yang and Lee (1952) have shown how the possible occurrence of phase transitions in a classical system of particles can be related to the behaviour of the zeros of the grand partition function Ξ in the complex plane of the fugacity variable z, in the limit where the size of the system becomes infinite. In this paper we shall consider one-dimensional continuum systems only, and we shall denote the grand partition function for such a system at fugacity z on a line of length L by $\Xi(z, L)$. The temperature $(k\beta)^{-1}$ is treated as a (positive) constant and is therefore not shown explicitly. A point z_0 in the complex z plane will be called a *limit point of zeros* of Ξ when the following condition is satisfied: for every neighbourhood N of z_0 and every number K, there exists a number $L > K$ and a z in N such that $\Xi(z, L) = 0$. An equivalent statement is this: z_0 is not a limit point of zeros of Ξ if and only if there exists some neighbourhood of z_0 that is free of zeros of $\Xi(z, L)$ for all sufficiently large L. We shall denote the set of limit points of zeros of Ξ by Z. The Yang–Lee theory shows that a phase transition can occur only at those values of z where Z meets the real positive z axis.

Up to now the set Z has been calculated directly only for lattice systems (Lee and Yang 1952, Hemmer *et al.* 1966, Nilsen and Hemmer 1967); Hemmer *et al.* also treat the continuum hard-rod system by making the lattice spacing tend to zero. In all these cases Z has been found to comprise a connected set of arcs in the z plane. It is the purpose of this paper to show rigorously that Z also consists of arcs in the case of one-dimensional continuum systems with hard cores and nearest-neighbour interactions, and to give an unambiguous prescription for determining these arcs from the functional relation giving the reduced pressure $\pi \equiv p/kT$ as a function of fugacity z when $z > 0$. This functional relation may be obtained either from the equation of state or from the formula

$$\pi(z) = \lim_{L\to\infty} L^{-1} \ln \Xi(z, L) \qquad \text{if } z > 0. \tag{1}$$

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Our prescription uses the complete analytic function $\Pi(z)$ obtained by analytic continuation of $\pi(z)$ into the complex z plane. Whenever one branch of $\Pi(z)$ is regular and has a larger real part than all the others, we call this branch $\Pi_{\max}(z)$. We shall show that Z comprises all the points where $\Pi_{\max}(z)$ does not exist, either because $\Pi(z)$ has a singularity (branch point) or because the branch of Π having largest real part is not unique.

Our prescription is closely related to the one used by Hemmer and his collaborators (Hiis Hauge and Hemmer 1963, Hemmer and Hiis Hauge 1964, Hemmer *et al.* 1966) to find Z for the one-dimensional gas of hard rods with or without a weak long-range interaction. The principle of their method (Byckling 1965) is to find a system of cuts in the complex z plane such that the real part of the analytic continuation of $\pi(z)$ into the cut plane is continuous across the cuts. A difficulty with their method is that there may be more than one system of cuts satisfying this condition. Our method differs from theirs in two ways: first, we prove (instead of assuming it) that $\pi(z)$ may be obtained by analytic continuation when z is complex, and that Z is a set of arcs; and secondly, the set of arcs we obtain is manifestly unique.

Our main results can be summarized in three theorems:

Theorem I

If $z \in$ G, where G is the set of values for which $\Pi_{\max}(z)$ exists, then

(*a*) $$\lim_{L\to\infty} L^{-1} \ln|\Xi(z, L)| = \operatorname{Re}\Pi_{\max}(z);$$

(*b*) $$z \notin \mathrm{Z}.$$

Theorem II

(*a*) The complement of G is composed of analytic arcs and

(*b*) G is simply connected.

Theorem III

(*a*) Z coincides with the complement of G, and

(*b*) if $N_{\mathrm{AB}}(L)$ is the number of zeros of $\Xi(z, L)$ in a region whose intersection with Z is the arc AB, then

$$\lim_{L\to\infty} \frac{N_{\mathrm{AB}}(L)}{L} = \frac{1}{2\pi}\int_{\mathrm{A}}^{\mathrm{B}} \frac{\partial(\Delta v)}{\partial s}\,\mathrm{d}s$$

where Δv is the discontinuity in $\operatorname{Im}\Pi_{\max}(z)$ across the arc AB and s is a parameter measuring distance along the arc.

A preliminary announcement of these results has already appeared (Elvey and Penrose 1968). Theorem I(*a*) was given there in the apparently stronger form

$$\lim L^{-1} \ln \Xi(z, L) = \Pi_{\max}$$

but there is no essential difference since the imaginary part of a logarithm can be defined uniquely only by an arbitrary convention.

2. The Laplace transform of the grand partition function

We consider a classical one-dimensional system of particles free to move on a line of length L. It is assumed that there are two-body forces only, with interaction potential $\varphi(r)$ satisfying

$$\begin{aligned} \varphi(r) &= +\infty \qquad \text{if } r < a \\ \varphi(r) &= 0 \qquad \text{if } r > 2a \end{aligned} \tag{2}$$

so that only nearest neighbours can interact. For $a \leqslant r \leqslant 2a$ the function $\varphi(r)$ is assumed to be real and of bounded variation.

The configurational integral for an n-particle system is

$$Q_n(L) \equiv \frac{1}{n!}\int_0^L \dots \int_0^L \mathrm{d}x_1 \dots \mathrm{d}x_n \exp\{-\beta U_n(x_1, \dots, x_n)\} \tag{3}$$

where $U_n(x_1, \dots, x_n)$ denotes the total potential energy of n particles at $x_1, \dots, x_n$. By virtue of (2) one can simplify (3) by using the symmetry of $U_n(x_1, \dots, x_n)$ in $x_1, \dots, x_n$, and specifying once and for all a particular ordering of the particles. One then obtains (for $n \geqslant 1$)

$$Q_n(L) = \int_0^L \mathrm{d}x_1 \int_{x_1}^L \mathrm{d}x_2 \dots \int_{x_{n-1}}^L \mathrm{d}x_n \prod_{t=1}^{n-1} \exp\{-\beta\varphi(x_{t+1}-x_t)\}. \tag{4}$$

This integral has the upper bound

$$Q_n(L) \leqslant (L^n/n!)\exp\{(n-1)\beta\Phi\} \tag{5}$$

where $\Phi \equiv -\inf \varphi(r)$. The grand partition function is defined by

$$\Xi(z, L) \equiv 1 + \sum_{n=1}^{\infty} z^n Q_n(L) \tag{6}$$

for all values, real or complex, of the fugacity z. According to (5) the series in (6) is absolutely convergent, and Ξ has the upper bound

$$|\Xi| \leqslant \exp(|z| L\, \mathrm{e}^{\beta\Phi}). \tag{7}$$

It was shown by Takahasi (1942, see also Gürsey 1950) that the Laplace transform of the configurational integral for the system considered here can be evaluated in a simple way. Here we shall take advantage of the further simplification that can be obtained (see Longuet-Higgins 1958) by using instead the Laplace transform of the grand partition function, which is defined by

$$\Upsilon(z, p) \equiv \int_0^\infty \mathrm{d}L\, \mathrm{e}^{-pL}\Xi(z, L) \tag{8}$$

provided that $\operatorname{Re} p$ exceeds the abscissa of convergence of the integral, defined by

$$A(z) \equiv \limsup_{L\to\infty} L^{-1} \ln|\Xi(z, L)|. \tag{9}$$

From the estimate (7) we see that

$$A(z) \leqslant |z|\, \mathrm{e}^{\beta\Phi}. \tag{10}$$

Since z is treated as a fixed parameter in this section, we shall abbreviate $\Upsilon(z, p)$ to $\Upsilon(p)$.

To calculate $\Upsilon(p)$ explicitly we substitute (6) into (8), obtaining

$$\Upsilon(p) = p^{-1} + \sum_{n=1}^{\infty} z^n \int_0^\infty \mathrm{d}L\, \mathrm{e}^{-pL} Q_n(L). \tag{11}$$

By (5), the series converges if $\operatorname{Re} p > |z| \exp(\beta\Phi)$, and the interchange of limits is justified by Lebesgue's theorem (Riesz and Nagy 1955, § 19). Using (4) we find, on changing to $L-x_n, x_n-x_{n-1}, \dots, x_2-x_1, x_1$ as variables of integration, that

$$\int_0^\infty \mathrm{d}L\, \mathrm{e}^{-pL} Q_n(L) = p^{-2}\{\psi(p)\}^{n-1} \tag{12}$$

where

$$\psi(p) = \int_0^\infty \mathrm{d}r\, \mathrm{e}^{-\beta\varphi(r)-pr}. \tag{13}$$

Unlike $\Upsilon(p)$, the function $\psi(p)$ does not depend on the parameter z. Substituting (12) into (11) we obtain

$$\Upsilon(p) = p^{-1}+zp^{-2}\{1-z\psi(p)\}^{-1} \tag{14}$$

for all p satisfying $\operatorname{Re} p > |z|\,e^{\beta\Phi}$.

To extend the definitions of $\psi(p)$ and $\Upsilon(p)$ into the rest of the p plane, we use (2) to write (13) in the form

$$\psi(p) = \int_a^{2a} dr\, e^{-\beta\varphi(r)-pr}+p^{-1}\,e^{-2ap}. \tag{15}$$

The integral is an entire analytic function of p; to prove this, we show that its first derivative

$$\lim_{h\to 0}\int_a^{2a} dr\, e^{-\beta\varphi(r)}(e^{-(p+h)r}-e^{-pr})/h \tag{16}$$

where $h \equiv |h|\,e^{i\theta}$, exists for all p and is independent of θ. This is done by subtracting

$$-\int_a^{2a} r\, dr \exp\{-\beta\varphi(r)-pr\}$$

the formal derivative of the integral in (15), from the expression whose limit is taken in (16), and noting that the absolute value of the resulting expression is at most

$$\lim \int_a^{2a} dr \left| e^{-\beta\varphi(r)-pr}\left(\frac{e^{-rh}-1}{h}+r\right)\right| \leqslant e^{\beta\Phi+2a|\operatorname{Re}p|}\lim\int_a^{2a} dr\, \tfrac{1}{2}r^2|h|\,e^{r|h|} = 0.$$

Thus the formula (15) provides the analytic continuation of the definition (13), which converges only for sufficiently large $\operatorname{Re} p$, into the whole p plane, and shows that $\psi(p)$ is a meromorphic function with just one simple pole at $p = 0$. Using the algebraic relation (14) we can now easily continue $\Upsilon(p)$ analytically into the whole p plane; it too is a meromorphic function.

3. The poles of the Laplace transform

To calculate $\pi(z)$, which depends on the behaviour of $\Xi(z, L)$ for large L, we shall apply the calculus of residues to the inversion integral for the Laplace transformation (8). This application depends on the positions of the poles of the function $\Upsilon(p)$. According to (14) these poles are the points $p_1, p_2, \ldots$ that satisfy the condition

$$\begin{aligned} \psi(p_i) &= 1/z \qquad \text{if } z \neq 0 \\ p_i &= 0 \qquad\quad \text{if } z = 0 \end{aligned} \tag{17}$$

(there is no pole at $p = 0$ if $z \neq 0$ since, by (15), $\psi = p^{-1}+O(1)$ for small p, so that (14) gives $\Upsilon(p) = \{1+O(z)\}/\{p-z+O(pz)\}$). The pole given by (17) is simple (Copson 1935, § 6.22) if $\psi'(p_i) \neq 0$, where $\psi' \equiv d\psi/dp$.

The function $\Upsilon(p)$ must have at least one pole, for if not the function $p\{\psi(p)-1/z\}$ would, by (15) and (17), be an entire function of p of order unity with no zeros and by Hadamard's factorization theorem (Copson 1935, § 7.6) would therefore have the form $C\exp(\alpha p)$, giving $\psi(p) = 1/z+C\exp(\alpha p)/p$ which is inconsistent with the consequence of (15) that $\lim_{p\to\infty}\psi(p) = 0$. (We are indebted to Professor W. Hayman for this argument.)

Only a finite number of the poles of $\Upsilon(p)$ lie to the right of any line $\operatorname{Re} p = \text{constant} = \lambda_0$, say. This is trivial if $\lambda_0 \geqslant A(z)$, since no poles at all lie in the region where the integral (8) converges. If $\lambda_0 < A(z)$ then all the poles in question lie in the strip $\lambda_0 < \operatorname{Re} p \leqslant A(z)$. We show that no poles can lie very far from the real p axis: integrating by parts in (15) we obtain

$$\left|\psi(p)-\frac{e^{-2ap}}{p}\right| = \left|\left[\frac{e^{-\beta\varphi(r)-pr}}{-p}\right]_{r=a}^{2a} - \int_a^{2a} \frac{e^{-\beta\varphi(r)-pr}\beta\,d\varphi(r)}{p}\right|$$
$$\leqslant \left\{2+\beta\int_a^{2a} |d\varphi(r)|\right\}\frac{e^{\beta\Phi+2a|\mathrm{Re}\,p|}}{|p|}. \tag{18}$$

Since φ is of bounded variation, it follows that $\psi(p) \to 0$, as $\mathrm{Im}\,p \to \pm\infty$ uniformly in the strip $\lambda_0 \leqslant \mathrm{Re}\,p \leqslant A(z)$, so that (by (17)) all the poles within this strip lie within a finite rectangle. Since $\Upsilon(p)$ is meromorphic, it can have only a finite number of poles in such a rectangle, which completes the proof.

A corollary of this result is that there must be a finite set of poles whose real parts are equal and exceed the real parts of all the other poles; we call the poles in this finite set the poles of *largest real part*. In general there is just one pole of largest real part, and that pole is simple; the set of all values of z for which this occurs will be denoted by G. (In § 1, a different definition of G was used, but lemma I will show that the two are equivalent.) The set of 'special' values of z for which this does not occur will be denoted by S.

We show next that every real non-negative value of z belongs to G. Equation (15) shows that, as p moves along the real axis from 0 to $+\infty$, the value of $\psi(p)$ decreases monotonically from $+\infty$ to 0. Consequently, if $z > 0$, there is just one real positive solution, say p_1, to the equation (17) for poles of $\Upsilon(p)$. Let p be any point with $\mathrm{Re}\,p \geqslant p_1$; then by (15) we have, firstly,

$$|\psi(p)| \leqslant \psi(\mathrm{Re}\,p)$$

with equality only if $\mathrm{Im}\,p = 0$, and secondly

$$\psi(\mathrm{Re}\,p) \leqslant \psi(p_1)$$

with equality only if $\mathrm{Re}\,p = \mathrm{Re}\,p_1$. Combining these two inequalities, we see that $|\psi(p)| < \psi(p_1)$ unless $p = p_1$, from which it follows, by (17), that if $p \neq p_1$ it cannot be a pole of $\Upsilon(p)$, so that p_1 is indeed the pole of largest real part.

Although we have so far treated z as a fixed parameter, it is important to know how the positions of the poles $p_1, p_2, \ldots$ depend on z. The following lemmas give useful information about this dependence. The first of them also serves to show that the definition of G used here is equivalent to the one used in § 1.

Lemma I

There exists a complete analytic function $\Pi(z)$ such that, for any complex z, the complex numbers $p_1, p_2, \ldots$ are the values of the various branches of $\Pi(z)$. The pole p_i is simple if, and only if, $\Pi(z)$ has no branch point when $\Pi(z) = p_i$.

Proof. Equation (15) shows that $\psi(p)$ has a simple pole at $p = 0$; therefore $1/\psi(p)$ has a simple zero at $p = 0$ and is analytic in a neighbourhood of $p = 0$. The relation $z = 1/\psi(p)$ can therefore be uniquely inverted near $p = 0$, to give p as a function of z. In conformity with the notation used in (17), we call this function $p_1(z)$, so that

$$\psi(p_1(z)) = 1/z \tag{19}$$

near $z = 0$. Now let $\Pi(z)$ be the complete analytic function (Ahlfors 1966, p. 276) obtained by analytic continuation from the function element $p_1(z)$; we shall show later that this definition of $\Pi(z)$ is equivalent to the one used in § 1. By the permanence of functional relations (Ahlfors 1966, p. 277) applied to (19), every branch of $\Pi(z)$ satisfies

$$\psi(\Pi(z)) = 1/z. \tag{20}$$

Consequently, by (17), every possible value of $\Pi(z)$ for a given value of z is a pole of $\Upsilon(p)$ for that value of z. Moreover, if p_i is such a possible value and $\Pi(z)$ has no branch point with $\Pi(z) = p_i$, then the relation (20) is uniquely soluble for $\Pi(z)$ close to p_i; this implies that $\psi'(p_i) \neq 0$ and hence, as shown in connection with (17), that the pole at p_i is simple. Conversely, if p_i is any simple pole of $\Upsilon(p)$, then we may join the point p_i to the origin of the p plane by a continuous path that avoids the isolated points where either $\psi(p)$ or $\mathrm{d}\psi/\mathrm{d}p$ vanishes; under the mapping $\psi(p) = 1/z$ this path has a unique continuous image in the z plane, and by continuing the function $p_1(z)$ analytically outwards along this path we obtain a branch of $\Pi(z)$ taking the value p_i for the given value of z. This completes the proof of lemma I.

Lemma II

Two branches of $\Pi(z)$ cannot have equal real parts throughout a region in the z plane.

Proof. Suppose, on the contrary, that two branches Π_1, Π_2 of $\Pi(z)$ satisfied the condition $\operatorname{Re}\Pi_1(z) = \operatorname{Re}\Pi_2(z)$ for all z in a region. Then the Cauchy–Riemann conditions would imply that the function $\Pi_1 - \Pi_2$ had a constant imaginary value, say $\mathrm{i}\omega$, for all z in this region. Since each of Π_1, Π_2 satisfies (20), we should have

$$\psi(\Pi_1(z)) = \psi(\Pi_1(z) - \mathrm{i}\omega)$$

for all z in the region, whence, by the permanence of functional relations,

$$\psi(p) = \psi(p - \mathrm{i}\omega)$$

holds for all p. But this cannot happen, for we have shown earlier in this section that $\psi(p) \to 0$ as $\operatorname{Im} p \to \pm\infty$. This completes the proof of lemma II.

4. Inversion of the Laplace transform

Since $\Xi(z, L)$ is a continuous function of L, the inversion formula for the Laplace transform (8) is (Widder 1941, pp. 37 and 63)

$$\Xi(z, L) = \frac{1}{2\pi\mathrm{i}} \lim_{M\to\infty} \int_{c-\mathrm{i}M}^{c+\mathrm{i}M} \mathrm{d}p\, \mathrm{e}^{pL}\Upsilon(p) \tag{21}$$

where c is any constant greater than $A(z)$. In the present section we shall use this formula to estimate $\Xi(z, L)$ for large L when $z \in \mathrm{G}$. Let the unique pole of $\Upsilon(p)$ with largest real part be p_1; then there exists a number λ such that $\lambda \neq 0$ and

$$p_i < \lambda < p_1 \tag{22}$$

for all the poles p_i of $\Upsilon(p)$ except the simple pole at p_1. Moving the contour of integration in (21) to the left and applying the calculus of residues at the simple pole p_1, we obtain

$$\Xi(z, L) = \exp(p_1 L)\left\{-\frac{1}{p_1{}^2\psi'(p_1)} + \epsilon(z, L)\right\} \tag{23}$$

where

$$\exp(p_1 L)\epsilon(z, L) \equiv \frac{1}{2\pi\mathrm{i}} \lim_{M\to\infty} \int_{\lambda-\mathrm{i}M}^{\lambda+\mathrm{i}M} \mathrm{d}p\, \mathrm{e}^{pL}\Upsilon(p). \tag{24}$$

We shall show that $\epsilon(z, L) \to 0$ as $L \to \infty$. Substituting from (14), with $p = \lambda + \mathrm{i}\mu$, we obtain

$$\epsilon(z, L) = \frac{\exp\{(\lambda - p_1)L\}}{2\pi}\left(\lim_{M\to\infty} \int_{-M}^{M} \frac{\mathrm{d}\mu\, \mathrm{e}^{\mathrm{i}\mu L}}{\lambda + \mathrm{i}\mu} + \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu\, z\, \mathrm{e}^{\mathrm{i}\mu L}}{(\lambda + \mathrm{i}\mu)^2\{1 - z\psi(\lambda + \mathrm{i}\mu)\}}\right). \tag{25}$$

The first integral in (25) is equal to $2\pi\,\mathrm{e}^{-\lambda L}$ if $\lambda > 0$ and to zero if $\lambda < 0$; its absolute value therefore cannot exceed 2π. To show that the second integral is also bounded in absolute value we note that, by (22) and (17), the function $z^{-1} - \psi(p)$ has no zeros on the line $\operatorname{Re} p = \lambda$; being analytic and therefore continuous, it is thus bounded away from zero on any closed segment of this line. Moreover, we have already shown (see equation (18)) that $\psi(\lambda+i\mu) \to 0$ as $\mu \to \pm\infty$; consequently $z^{-1}-\psi(p)$ is bounded away from zero on the entire line $\operatorname{Re} p = \lambda$. We can therefore find a number α satisfying $0 < \alpha \leqslant |z^{-1}-\psi(\lambda+i\mu)|$ for all real μ, and so the absolute value of the second integral in (25) cannot exceed $\int d\mu\, |z|/(\lambda^2+\mu^2)\alpha = \pi|z|/\lambda\alpha$. Both integrals in (25) being bounded, it follows by (22) that

$$\lim_{L\to\infty} \epsilon(z, L) = 0. \tag{26}$$

Combining (26) with (23) and using lemma I, we find that

$$\lim_{L\to\infty} L^{-1} \ln|\Xi(z, L)| = \operatorname{Re}\Pi_{\max}(z) \qquad \text{if } z \in \mathrm{G} \tag{27}$$

where $\Pi_{\max}(z)$ is the branch of $\Pi(z)$ having largest real part; by lemma I this branch is unique for all $z \in \mathrm{G}$. For real positive values of z, which were shown in § 3 all to belong to G, $\Xi(z, L)$ and $p_1(z)$ are also real and positive, so that (27) may be strengthened to

$$\pi(z) = \Pi_{\max}(z) \qquad \text{if } z > 0 \tag{28}$$

where $\pi(z)$ is defined in (1). This result shows that the functions denoted by $\Pi_{\max}(z)$ in this section and in § 1 are identical; the result (27) is thus identical with theorem I(a).

To prove theorem I(b) we must show that every point of G has a neighbourhood that is free of zeros of Ξ for all sufficiently large L. Let z_0 be any point of G, and choose λ so that (22) holds when $z = z_0$; then, since all branches of $\Pi(z)$ are analytic and therefore continuous, it follows by lemma I that (22) also holds throughout some neighbourhood N of z_0. Moreover, since the pole at $p_1(z_0)$ is simple, we have $\psi'(p_1(z_0)) \neq 0$, and hence N may be chosen so that $\psi'(p_1(z)) \neq 0$ throughout N. These two conditions ensure that N is a subset of G. Let C be any compact subset of N. We shall show, using (23), that $\Xi(z, L)$ has no zeros in C when L is large. Since $\psi'(p)$ and $p_1(z)$ are continuous functions, the term $-1/p_1^2\psi'(p_1)$ is bounded away from zero for $z \in \mathrm{C}$; therefore it is sufficient to show that $\epsilon(z, L) \to 0$ uniformly on C. By the definition of N there is a positive lower bound, say δ, on the values taken by $\operatorname{Re} p_1(z) - \lambda$ for $z \in \mathrm{C}$; consequently the exponential in (25) is bounded above by the function $\exp(-L\delta)$. The factor multiplying this exponential is also uniformly bounded, this time by a constant: as we have already seen, the first integral in brackets cannot exceed 2π and the second cannot exceed $\pi|z|/\lambda\alpha'$, where α' is a positive lower bound on $|z^{-1}-\psi(\lambda+i\mu)|$ for all z in C and all real μ. (That such a lower bound exists follows from an argument similar to the one used in proving (26), with $z^{-1}-\psi(\lambda+i\mu)$ now treated as a function of the three variables $\operatorname{Re} z$, $\operatorname{Im} z$ and μ which is continuous and has no zeros when $z \in \mathrm{C}$ and μ lies in an arbitrary closed interval.) This completes the proof that $\epsilon(z, L) \to 0$ uniformly on C. Using this result in (23), we conclude that C is free of zeros of $\Xi(z, L)$ for all sufficiently large L. Choosing C to be the closure of some neighbourhood of z_0, we see by the definition given in § 1 that z_0 cannot be a limit point of zeros of Ξ. This completes the proof of theorem I.

5. Proof that the limit points lie on arcs

Having shown that all the limit points of zeros of $\Xi(z, L)$ belong to S, the point set in the z plane complementary to G, we have still to determine which points of S are limit points of $\Xi(z, L)$ and to obtain the limiting density of the zeros on S. As a first step we show that S is the union of a set of analytic arcs. Let z_0 be any point of S, and let $p_1, \ldots, p_k$ be the poles of largest real part and $p_{k+1}, p_{k+2}, \ldots$ be the remaining poles for the function $\Upsilon(p)$ discussed in § 3, when the parameter z takes the value z_0. It is then possible to choose a real number λ such that

$$p_i > \lambda > p_j \qquad \text{if } i \leqslant k \text{ and } j \geqslant k+1. \tag{29}$$

Let the order of the pole p_i be O_i ($i = 1, \ldots, k$), then by the definitions of z_0 and of S we have $\mathrm{O}_1+\mathrm{O}_2+\ldots+\mathrm{O}_k \geqslant 2$. According to lemma I, if p_i is a simple pole (i.e. if $\mathrm{O}_i = 1$) then the complete analytic function Π has a branch that is analytic near z_0 and takes the value p_i when $z = z_0$. We denote this branch by $\Pi_{i,1}$. If, on the other hand, $O_i \geqslant 2$, then (by (14) and (20)) Π has an algebraic branch point of order $O_i - 1$ at z_0, and at any point z sufficiently close to z_0 it has O_i branches all of which tend to the value p_i as $z \to z_0$. We denote these branches by $\Pi_{i,1}(z), \Pi_{i,2}(z), \ldots, \Pi_{i,O_i}(z)$.

Since all branches of $\Pi(z)$ are continuous in the limit $z \to z_0$, we deduce from (29) the existence of a neighbourhood N of z_0 such that every point z in N has the following two properties:

(i) the functions $\Pi_{i,m}(z)$ $(m = 1, \ldots, O_i;\ i = 1, \ldots, k)$ are analytic at z if $z \neq z_0$, and

(ii)
$$\operatorname{Re} \Pi_{i,m}(z) > \lambda > \operatorname{Re} \Pi_j(z) \tag{30}$$
$$(i = 1, \ldots, k;\ m = 1, \ldots, O_i;\ j = k+1, k+2, \ldots).$$

The property (ii) shows, by lemma I, that the pole or poles of largest real part for any z in N must come from the set of numbers $\Pi_{i,m}(z)$ with $i = 1, \ldots, k$ and $m = 1, \ldots, O_i$. By (i), the relevant poles are all simple if $z \neq z_0$, so that the only points of S in N are points for which two or more branches have equal real parts:

$$\operatorname{Re}[\Pi_{i,m}(z) - \Pi_{i',m'}(z)] = 0 \tag{31}$$

with $1 \leqslant i \leqslant k$, $1 \leqslant i' \leqslant k$, and $m \neq m'$ if $i = i'$.

Equation (31) defines a subset of N which we denote by $H(i, m; i', m')$. This subset is a union of analytic arcs: let us denote the analytic function in square brackets by $\chi(z)$; we know from lemma II that $\chi(z)$ cannot be a constant, so its derivative can vanish only at a finite number of points of N, and at all other points the arcs may be parametrized by the equation $z = \chi^{-1}(it)$, where $t \equiv \operatorname{Im}[\Pi_{i',m'} - \Pi_{i,m}]$ and χ^{-1}, the inverse of the function χ, is analytic because $d\chi/dz \neq 0$.

The part of S that lies within N consists of the point z_0 and the parts of the point sets $H(i, m; i', m')$ for which the additional conditions

$$\begin{aligned} \operatorname{Re}[\Pi_{i,m}(z) - \Pi_{i'',m''}(z)] &\geqslant 0 \\ \operatorname{Re}[\Pi_{i',m'}(z) - \Pi_{i'',m''}(z)] &\geqslant 0 \end{aligned} \tag{32}$$

are satisfied for all $i'' = 1, 2, \ldots, k$ and $m'' = 1, 2, \ldots, O_{i''}$. Since all branches of Π are continuous, the part of $H(i, m; i', m')$ consistent with (32) comprises arcs that are subsets of the arcs constituting $H(i, m; i', m')$; their end points are intersections of $H(i, m; i', m')$ with either $H(i, m; i'', m'')$ or $H(i', m'; i'', m'')$. Thus the part of S within N is the union of a finite number of analytic arcs and isolated points. Since z_0 is an arbitrary point of S, we conclude that the whole of S is the union of a set of analytic arcs and isolated points, the isolated points and the end points of the arcs having no finite limit point (since every point z has a neighbourhood that contains, if $z \in S$, only a finite number of such isolated points and end points or, if $z \in G$, none at all).

To complete the proof of theorem II(a) we must show that there are no isolated points in S: this will be possible as soon as we have proved theorem II(b), that G is simply connected. To prove theorem II(b) we show that any smooth Jordan curve Γ in G has no points of S inside it. The proof depends on the fact that the continuous real-valued function

$$u(z) \equiv \max_i \operatorname{Re} \Pi_i(z) \tag{33}$$

is subharmonic (Ahlfors 1966, p. 237, Rado 1949). This fact is obvious when $z \in G$, since $\Pi_{\max}(z)$ is analytic and therefore its real part is harmonic. When $z \in S$ we use the expansion of $\Pi_{i,m}(\zeta)$ in fractional powers of $\zeta - z$ (Ahlfors 1966, p. 290):

$$\Pi_{i,m}(\zeta) = \Pi_i(z) + \sum_{n=1}^{\infty} a_{i,n}(\zeta - z)^{n/O_i} \exp(2\pi i n m/O_i)$$

from which it follows that, for small enough values of ρ,

$$\frac{1}{2\pi}\int_0^{2\pi} u(z+\rho\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \geqslant \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,\frac{1}{k}\sum_{i=1}^{k}\frac{1}{\mathrm{O}_i}\sum_{m=1}^{\mathrm{O}_i}\Pi_{i,m}(z+\rho\,\mathrm{e}^{\mathrm{i}\theta})$$

$$= \mathrm{Re}\,\frac{1}{k}\sum_{i=1}^{k}\Pi_i(z) = u(z)$$

in agreement with the definition of a subharmonic function.

It is a property of subharmonic functions (see Rado 1949, § 4.31) that

$$\int_\Gamma \frac{\partial u}{\partial n}\,\mathrm{d}s \geqslant 0 \tag{34}$$

where $\mathrm{d}s$ is the element of arc length on Γ and $\partial u/\partial n$ is the derivative of u taken in a direction normal to Γ, the positive sense being outwards. The two sides of (34) are equal only if u is harmonic inside Γ. By the definition of Γ, however, $\Pi_{\max}$ is analytic at every point of Γ; applying the Cauchy–Riemann conditions we therefore have

$$\int_\Gamma \frac{\partial\,\mathrm{Re}\,\Pi_{\max}}{\partial n}\,\mathrm{d}s = \int_\Gamma \frac{\partial\,\mathrm{Im}\,\Pi_{\max}}{\partial s}\,\mathrm{d}s = 0 \tag{35}$$

since $\Pi_{\max}$ is single-valued. Here $\partial/\partial s$ means a derivative in the direction of Γ. Combining (34) and (35) we see that u is indeed harmonic inside Γ, and therefore $\Pi_{\max}$ is analytic inside Γ, which implies that every point inside Γ is a point of G. This completes the proof of theorem II(b), that G is simply connected, and the fact that S can have no isolated points follows at once, completing the proof of theorem II(a), that S consists of analytic arcs.

6. The limit points of zeros

In this section we show that every point of S is a limit point of zeros of $\Xi(z, L)$. As in the previous section we denote the point in question by z_0 and suppose, for the moment, that $\Xi(z_0, L) \neq 0$. By Jensen's formula (Ahlfors 1966, p. 206) we then have, for any positive number ρ,

$$\int_0^\rho N(r, z_0, L)\,\mathrm{d}r/r = LI(\rho, z_0, L)/2\pi - \ln|\Xi(z_0, L)| \tag{36}$$

where $N(r, z_0, L)$ denotes the number of zeros of $\Xi(z, L)$ inside the circle $|z-z_0| = r$, and

$$I(\rho, z_0, L) \equiv \int_0^{2\pi} L^{-1}\ln|\Xi(z_0+\rho\,\mathrm{e}^{\mathrm{i}\theta}, L)|\,\mathrm{d}\theta. \tag{37}$$

By (27) and (33), the integrand in (37) tends to the limit $u(z_0+\rho\exp(\mathrm{i}\theta))$ as $L\to\infty$, provided that $z_0+\rho\exp(\mathrm{i}\theta)\in \mathrm{G}$. Apart from the highly exceptional case where one of the arcs constituting S is also an arc of the circle $|z-z_0| = \rho$, the set S intersects this circle at only a finite number of points, and therefore almost all points of this circle belong to G. This will enable us to prove that

$$\lim_{L\to\infty} I(\rho, z_0, L) = \int_0^{2\pi} u(z(\theta))\,\mathrm{d}\theta \tag{38}$$

where $z(\theta) \equiv z_0+\rho\exp(\mathrm{i}\theta)$.

Let δ be any small positive number and let $\mathrm{C}(\delta)$ and $\mathrm{C}'(\delta)$ be the parts of the circle $|z-z_0| = \rho$ whose distances from S are, respectively, not less than δ and less than δ. We first show that $\mathrm{C}(\delta)$ contributes equally to both sides of (38). By Lebesgue's theorem (Riesz and Nagy 1955, § 19) it is sufficient to show that $L^{-1}\ln|\Xi(z(\theta), L)|$ is bounded above and below, uniformly in L, for $z(\theta)\in\mathrm{C}(\delta)$. By (7) an upper bound is

$$L^{-1}\ln|\Xi(z(\theta), L)| \leqslant |z(\theta)|\,\mathrm{e}^{\beta\Phi} \leqslant (|z_0|+\rho)\,\mathrm{e}^{\beta\Phi}. \tag{39}$$

For a lower bound we use Yang and Lee's (1952) factorization in the form

$$L^{-1}\ln|\Xi(z,L)| = L^{-1}\sum_i \ln|1-z/z_i(L)| \tag{40}$$

where $z_1(L)$, $z_2(L)$, ... are the zeros of $\Xi(z, L)$. Since all the limit points of zeros are at least δ distant from all points of $C(\delta)$, the region consisting of all points less than $\frac{1}{2}\delta$ distant from $C(\delta)$ must be free of zeros for all sufficiently large L (for otherwise it would contain a limit point of zeros, by the Bolzano–Weierstrass theorem (Ahlfors 1966, p. 63), but the definitions imply that all such limit points are at least δ distant from C). Since

$$\left|1-\frac{z}{\zeta}\right| \geqslant \frac{|\zeta-z|}{|\zeta-z|+|z|} \tag{41}$$

and the number of zeros of $\Xi(z, L)$ is at most $L(a^{-1}+L^{-1})$ because of the hard cores, we see from (40) that

$$\begin{aligned} L^{-1}\ln|\Xi(z(\theta),L)| &\geqslant (a^{-1}+L^{-1})\ln\frac{\frac{1}{2}\delta}{\frac{1}{2}\delta+|z(\theta)|} \\ &\geqslant (a^{-1}+L^{-1})\ln\frac{\frac{1}{2}\delta}{\frac{1}{2}\delta+|z_0|+\rho} \end{aligned} \tag{42}$$

for all sufficiently large L. It follows, by Lebesgue's theorem and equations (27) and (33), that

$$\lim_{L\to\infty}\int_{z(\theta)\in C(\delta)} L^{-1}\ln|\Xi(z(\theta),L)|\,\mathrm{d}\theta = \int_{z(\theta)\in C(\delta)} u(z(\theta))\,\mathrm{d}\theta. \tag{43}$$

To complete the proof we show that the contribution of $C'(\delta)$ to the two sides of (38) tends to zero with δ. Since C' is the union of a finite number of arcs of the circle $|z-z_0| = \rho$, it is sufficient to consider just one of these arcs: let it comprise the points $z_0+\rho\exp(\mathrm{i}\theta)$ with $\theta_1 < \theta < \theta_2$. Since δ is small we may assume $\theta_2-\theta_1 < \pi$. Applying (39) and (40) we have

$$\int_{\theta_1}^{\theta_2} L^{-1}\ln|\Xi(z(\theta),L)|\,\mathrm{d}\theta \leqslant (\theta_2-\theta_1)(|z_0|+\rho)\,\mathrm{e}^{\beta\Phi} \tag{44}$$

and

$$\geqslant (a^{-1}+L^{-1})\min_\zeta\int_{\theta_1}^{\theta_2}\ln|1-z(\theta)/\zeta|\,\mathrm{d}\theta. \tag{45}$$

Writing $\zeta \equiv z_0+\sigma\,\mathrm{e}^{\mathrm{i}\tau}$ where $\sigma \geqslant 0$, we have

$$\int_{\theta_1}^{\theta_2}\ln|1-z(\theta)/\zeta|\,\mathrm{d}\theta = \int_{\theta_1-\tau}^{\theta_2-\tau}\ln\frac{\rho|\sigma/\rho-\mathrm{e}^{\mathrm{i}\phi}|}{|z_0+\sigma\,\mathrm{e}^{\mathrm{i}\tau}|}\,\mathrm{d}\phi \tag{46}$$

where $\phi \equiv \theta-\tau$. If $\sigma/\rho < 1$ the right-hand side of (46) has the lower bound

$$\int_{\theta_1-\tau}^{\theta_2-\tau}\ln\frac{\rho|\sin\phi|}{|z_0|+\rho}\,\mathrm{d}\phi \tag{47}$$

since the distance from the point $\mathrm{e}^{\mathrm{i}\phi}$ to the real axis is $\sin\phi$. If $\sigma/\rho > 1$ we write the right-hand side of (46) in the form

$$\int_{\theta_1-\tau}^{\theta_2-\tau}\ln\frac{\rho|\mathrm{e}^{-\mathrm{i}\phi}-\rho/\sigma|}{|z_0\rho/\sigma+\rho\,\mathrm{e}^{\mathrm{i}\tau}|}\,\mathrm{d}\phi \tag{48}$$

whence it follows that the lower bound (47) still applies. Regarded as a function of τ, the integral (47) takes its minimum value when $\tau = \frac{1}{2}(\theta_1+\theta_2)$, and therefore (45) implies

$$\int_{\theta_1}^{\theta_2} L^{-1} \ln|\Xi(z(\theta), L)| \, d\theta \geqslant 2(a^{-1}+L^{-1}) \int_0^{(\theta_2-\theta_1)/2} \ln \frac{\rho|\sin\phi|}{|z_0|+\rho} d\phi. \tag{49}$$

As $\delta \to 0$, the angular arc length $\theta_2-\theta_1 \to 0$, and the upper and lower bounds in (44) and (49) both tend to zero uniformly in L. Summing the contributions of the finite set of arcs constituting $C'(\delta)$, we conclude that

$$\lim_{\delta\to 0}\left(\lim_{L\to\infty} \int_{z(\theta)\in C'(\delta)} L^{-1} \ln|\Xi| \, d\theta - \int_{z(\theta)\in C'(\delta)} u \, d\theta\right) = 0. \tag{50}$$

The analogue of this equation with C replacing C′ is also true, by (43). Adding the two equations, and noting that since $C(\delta)+C'(\delta)$ is the whole circle the limit $\delta \to 0$ is now superfluous, we complete the proof of (38).

To apply (38), let ρ_1 and ρ_2 be any two numbers such that $0 < \rho_1 < \rho_2$ and no arc of S is an arc of either of the circles $|z-z_0| = \rho_1, \rho_2$. Giving ρ the values ρ_1 and ρ_2 in (36), subtracting the resulting equations, dividing by L and taking the limit $L \to \infty$, we obtain (using (38))

$$\lim_{L\to\infty} \int_{\rho_1}^{\rho_2} \frac{N(r, z_0, L)}{L} \frac{dr}{r} = \frac{1}{2\pi} \int_0^{2\pi} \{u(z_2(\theta)) - u(z_1(\theta))\} \, d\theta \tag{51}$$

where $z_1(\theta) = z_0+\rho_1 e^{i\theta}$, etc. Since $N(r, z_0, L)$ is a non-decreasing function of r, we deduce from (51), after dividing both sides by $\rho_2-\rho_1$, that

$$\limsup_{L\to\infty} \frac{N(\rho_1, z_0, L)}{L\rho_2} \leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{u(z_0+\rho_2 e^{i\theta}) - u(z_0+\rho_1 e^{i\theta})}{\rho_2-\rho_1} d\theta. \tag{52}$$

Taking the limit $\rho_2 \to \rho_1$, and writing ρ in place of ρ_1, we obtain

$$\limsup_{L\to\infty} \frac{N(\rho, z_0, L)}{L} \leqslant \frac{\rho}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial\rho}\{u(z_0+\rho e^{i\theta})\} \, d\theta \tag{53}$$

provided that $\Pi_{max}(z)$ has no branch point on the circle $|z-z_0| = \rho$ (this condition ensures that $\partial u/\partial\rho$ is uniformly continuous in the annulus $\rho_1 \leqslant |z-z_0| \leqslant \rho_2$). In a similar way we can show that the right-hand side of (52) is a lower bound on $\liminf N(\rho_2, z_0, L)/L\rho_1$ and hence that the right-hand side of (53) is a lower bound on $\liminf N(\rho, z_0, L)/L$; consequently we deduce that $\lim_{L\to\infty} N(\rho, z_0, L)/L$ exists and is given by

$$\lim_{L\to\infty} \frac{N(\rho, z_0, L)}{L} = \frac{1}{2\pi} \oint \frac{\partial u}{\partial\rho} ds \tag{54}$$

where $ds = \rho \, d\theta$ is the element of arc length on the circular contour of integration $|z-z_0| = \rho$. By the Cauchy–Riemann condition this result may also be written

$$\lim_{L\to\infty} \frac{N(\rho, z_0, L)}{L} = \frac{1}{2\pi} \oint \frac{\partial v}{\partial s} ds \tag{55}$$

where $v \equiv \operatorname{Im} \Pi_{max}$ and

$$\frac{\partial v}{\partial s} \equiv \rho^{-1} \frac{\partial}{\partial\theta}\{v(z_0+\rho e^{i\theta})\}$$

for $z_0+\rho e^{i\theta} \in G$.

In order to prove that z_0 is a limit point of zeros it is sufficient to show that the right-hand side of (55) is non-zero for every sufficiently small ρ. If z_0 is an interior point of one of the arcs constituting S, we choose ρ so small that the only part of S inside the circle of integration is the part of this arc near z_0. Since the integrand in (55) is an exact differential except at the two points of intersection (say A and B) of the contour of integration with this arc, we may deform the contour so that it goes from A to B on one side of the arc and back again on the other; this gives

$$\lim_{L\to\infty} \frac{N(\rho, z_0, L)}{L} = \frac{1}{2\pi}\int_A^B \frac{\partial(\Delta v)}{\partial s}\,\mathrm{d}s \tag{56}$$

where Δv is the discontinuity in v across the arc. The formula (56) is equivalent to one first given by Hemmer and Hiis Hauge (1964), who obtained it by an electrostatic analogy. From it, theorem III(b) follows at once.

To prove theorem III(a), that $z_0 \in \mathrm{S}$ is a limit point of zeros, we use lemma II, which implies that the two branches of $\Pi(z)$ whose real parts are jointly maximal on the arc through z_0 cannot differ by a constant; consequently $\partial(\Delta v)/\partial s$ is not identically zero near z_0, and we conclude that the right-hand side of (56) is non-zero for arbitrarily small ρ and hence that z_0 is a limit point of zeros of $\Xi(z, L)$. If z_0 is an end point, or a point of intersection, of the arcs constituting S, a similar argument may be used, or we may rely on the observation that such an end point, being a limit point of limit points of zeros, must itself be a limit point of zeros. This completes the proof of theorem III.

7. The hard-rod system

As an illustration, we use theorem III to determine the limit points of zeros for a system of hard rods, for which

$$\begin{aligned} \varphi(r) &= +\infty \qquad \text{if } r < a \\ \varphi(r) &= 0 \qquad \text{if } r > a. \end{aligned} \tag{57}$$

Equation (13) gives here

$$\psi(p) = \mathrm{e}^{-ap}/p \tag{58}$$

so that (20) becomes

$$\Pi(z)\exp\{a\Pi(z)\} = z. \tag{59}$$

We show first that all the limit points of zeros lie on the real axis. By theorem III(a), the set Z of limit points comprises those values of z for which either (i) $\Pi(z)$ has two branches of largest real part or (ii) $\Pi(z)$ has an algebraic branch point of largest real part. In case (i) let the two branches with largest real part at z be Π_1 and Π_2, so that $\mathrm{Re}\,\Pi_1(z) = \mathrm{Re}\,\Pi_2(z)$. Taking the modulus of both sides of (59), we have

$$\{\mathrm{Re}\,\Pi(z)\}^2 + \{\mathrm{Im}\,\Pi(z)\}^2 = |z|^2\exp\{-2a\,\mathrm{Re}\,\Pi(z)\} \tag{60}$$

which implies (since $\Pi_1 \neq \Pi_2$) that $\mathrm{Im}\,\Pi_1 = -\mathrm{Im}\,\Pi_2 \neq 0$. Thus Π_1 and Π_2 are complex conjugates, and by substituting both of them into (59) we see that z is its own complex conjugate, i.e. z is real. To treat case (ii) we differentiate (59) and set $\mathrm{d}z/\mathrm{d}\Pi = 0$; this shows that the only algebraic branch point is

$$\Pi(z) = -1/a, \qquad z = -1/ea \tag{61}$$

so that z is again real.

To discover which parts of the real axis constitute the set Z, we consider separately the segments $-\infty < z \leqslant -1/ea$, $-1/ea < z < 0$ and $0 \leqslant z < \infty$. For the segment $-\infty < z \leqslant -1/ea$, we note that as the real variable t goes from $-\infty$ to $+\infty$ the function $t\,\mathrm{e}^{at}$ decreases from 0 to $-1/ea$ (reached when $t = -1/a$) and then increases to $+\infty$. Consequently, when $z < -1/ea$ equation (59) has no real solution $\Pi(z)$ and, since the complex solutions occur in conjugate pairs, there must be at least two branches of $\Pi(z)$ having largest real part; it follows by theorem III that if $z < -1/ea$ then z belongs to Z. The end point of this segment, $z = -1/ea$, also belongs to Z since a limit point of limit points of zeros is itself a limit point of zeros.

In the next case, $-1/ea < z < 0$, the equation $t\,e^{at} = z$ has two real roots, the larger of which (say $t(z)$) lies in the range $-1/a < t < 0$. We shall show that any complex solution of (59), say $\Pi(z) \equiv \lambda + i\mu$, has a smaller real part than $t(z)$. Taking the imaginary part of (59) we find (since z is real and $\mu \neq 0$) that $\lambda = -\mu \cot a\mu$. Using this in the real part of (59) we obtain

$$-z\frac{\sin a\mu}{\mu} = e^{a\lambda}. \tag{62}$$

Since $-1 \leqslant (\sin a\mu)/a\mu \leqslant 1$, the left-hand side of (62) cannot exceed $1/e$, and so (62) implies $\lambda \leqslant -1/a$. Since $-1/a < t$, we see that $\Pi(z) = t(z)$ is the solution of (59) having largest real part; since this solution is unique, the segment $-1/ea < z < 0$ does not intersect Z, by theorem I(b).

For the remaining segment, $0 \leqslant z < \infty$, we use the result proved in § 3, that every real non-negative value of z belongs to G, which shows that this segment too does not intersect Z. We conclude, therefore, that for the hard-rod system Z comprises all points with $z \leqslant -1/ea$ and no others. This confirms the result obtained by Hemmer and Hiis Hauge (1964) and by Hemmer *et al.* (1966) using less rigorous methods.

8. Discussion

As indicated in the introduction, our work owes much to that done by Hemmer in collaboration with Hiis Hauge and Aasen, and to that of Byckling (1965). Apart from the greater rigour of our work, the main new result is to show (theorem I) how the criterion of *largest real part* serves to single out the branch of the complete analytic function $\Pi(z)$ that is relevant to the behaviour of $\Xi(z, L)$ for large L, and (theorem III) that the limit points of zeros are the values of z for which two or more branches have largest real part, or there is a branch point of largest real part. The corresponding criterion for a class of lattice systems was given by Nilsen and Hemmer (1967). Both criteria are natural generalizations of criteria that had been used earlier for real positive z: Kac (1959), in his study of a system of hard rods with exponentially decreasing forces, used the fact (obtainable from (1) and (9)) that $\pi = p/kT$ is equal to the abscissa of convergence of the Laplace transform (8); and many problems in lattice statistics are equivalent to finding the eigenvalue of a matrix that has largest modulus (so that its logarithm has largest real part), as shown first by Kramers and Wannier (1941). van Hove (1950), Edwards and Lenard (1962) and Baxter (1964, 1965) have also used this type of criterion.

Possibly the theorems proved here could be extended to more general classes of systems. The most immediate generalization would be to a system of hard rods with forces extending to second-nearest neighbours or, better still, to forces of infinite range such as the exponential force considered by Kac (1959). Another possible generalization is to consider complex values for $\beta \equiv 1/kT$ as well as z; aspects of the behaviour of Ξ in the complex β plane have been considered by Fisher (1965) and by Jones (1966). Some of our theorems generalize easily to the β plane, but we have not succeeded in generalizing theorem III.

It was shown by van Hove (1950) that the system studied here has no phase transition. Our results confirm this, since phase transitions occur only where Z intersects the positive real axis but, as shown in § 3, the positive real axis lies entirely within G. The most important question raised by our work is whether the 'principle of largest real part' embodied in theorem I can be extended to systems that do have phase transitions. There is reason to believe (Elvey 1968) that theorem I does not hold for a system obeying van der Waals' equation of state, but the general question is still open.

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